Math 40960, Topics in Geometry

Problem Set 2, due March 21, 2014

Note: Answer the questions as I've written them here; the references to the books are just so you know where they came from. Of course you should show all your work. And as always, if you get any help from any source, in person or online or otherwise, you need to acknowledge it.

1. Let $\mathbb{Z}_5$ be the field of order 5. So as a set, $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. Just to remind you, addition gives $\mathbb{Z}_5$ the structure of a group, with group table

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

From this field we obtain the affine plane $\mathbb{A}^2(\mathbb{Z}_5)$, as discussed in class. So it has order $n = 5$, and the plane has $5^2 = 25$ elements. Let's see how to obtain the associated $n - 1 = 4$ Latin squares.

Let $\mathcal{P}_1$ be the pencil of lines of the form $y = a$, where $a \in \mathbb{Z}_5$. (This corresponds to the lines ℓ_i in class.) Let $\mathcal{P}_2$ be the pencil of lines of the form $x = b$, where $b \in \mathbb{Z}_5$. (This corresponds to the lines m_i in class.) An element $(i, j) \in \mathbb{A}^2(\mathbb{Z}_5)$ is thus the intersection of the lines $y = i$ (from $\mathcal{P}_1$) and $x = j$ (from $\mathcal{P}_2$).

(a) Verify that every other line in $\mathbb{A}^2(\mathbb{Z}_5)$ is of the form $y = mx + k$, where $m \in \{1, 2, 3, 4\}$ and $k \in \{0, 1, 2, 3, 4\}$.

(b) Show that the four remaining pencils (other than $\mathcal{P}_1$ and $\mathcal{P}_2$) are defined by the equations

$$\begin{aligned} \mathcal{Q}_1 &= \text{the lines } y = x + k \text{ where } k \in \{0, 1, 2, 3, 4\}; \\ \mathcal{Q}_2 &= \text{the lines } y = 2x + k \text{ where } k \in \{0, 1, 2, 3, 4\}; \\ \mathcal{Q}_3 &= \text{the lines } y = 3x + k \text{ where } k \in \{0, 1, 2, 3, 4\}; \\ \mathcal{Q}_4 &= \text{the lines } y = 4x + k \text{ where } k \in \{0, 1, 2, 3, 4\}; \end{aligned}$$

(c) Let's construct the Latin square corresponding to $\mathcal{Q}_2$. Label the lines in $\mathcal{Q}_2$ in the natural way, by assigning to $y = 2x + 0$ the label "0," assigning to $y = 2x + 1$ the label "1," etc. With this notation, find the Latin square corresponding to $\mathcal{Q}_2$.

2. Compute the Euler characteristic of the simplicial complex

$$\Delta = \langle 123, 134, 145, 125, 236, 346, 456, 256 \rangle.$$

3. (This is taken almost verbatim from Hartshorne p. 72.)

Kirkman's schoolgirl problem (1850) is as follows. In a certain school there are 15 girls. It is desired to make a seven-day schedule such that each day the girls can walk in the garden in five groups of three, in such a way that each girl will be in the same group with each other girl exactly once in the week. How should the groups be formed each day?

To make this into a geometry problem, think of the girls as points, think of the groups of three as lines, and think of each day as describing a set of five lines, which we call a pencil. A *Kirkman geometry* consists of a set, whose elements are *points*, together with certain subsets called *lines*, and certain collections of lines called *pencils*, satisfying the following axioms:

(**K1**) Two distinct points lie on a unique line.

(**K2**) All lines contain the same number of points.

(**K3**) There exist three non collinear points.

(**K4**) Each line is contained in a unique pencil.

(**K5**) Each pencil consists of a set of parallel lines whose union is the whole set of points.

(a) Show that any affine plane gives a Kirkman geometry, where we take the pencils to be the set of all lines parallel to a given line. Feel free to quote facts from class.

(b) By part (a) we know that there exist Kirkman geometries with 4, 9, 16, 25, ... points. (You don't have to justify this fact.) Suppose you could find a Kirkman geometry with 15 points. Carefully explain how this would give a solution of the original schoolgirl problem. This includes the following assertions that should be included and proven in your answer:

(i) Each line contains three points.
(ii) Each pencil contains five lines.
(iii) There are seven pencils.

To clarify: I'm saying that you should show that if there exists a Kirkman geometry with 15 points then (i), (ii) and (iii) must hold. Then *apply* this fact to the schoolgirls.

(c) In fact there **is** a Kirkman geometry with 15 points (you do not have to find it). On the other hand there is no affine plane with 15 points (again this is a fact that you do not have to prove). So it's obvious that the converse of (a) is not true: a Kirkman geometry is not necessarily an affine plane (again you don't have to explain this since it's obvious from the previous two sentences). Here is what I would like you to do: compare the axioms of an affine plane (using Moorhouse's axioms) with those of a Kirkman geometry to explain why not all the axioms of an affine plane follow from those of a Kirkman geometry.

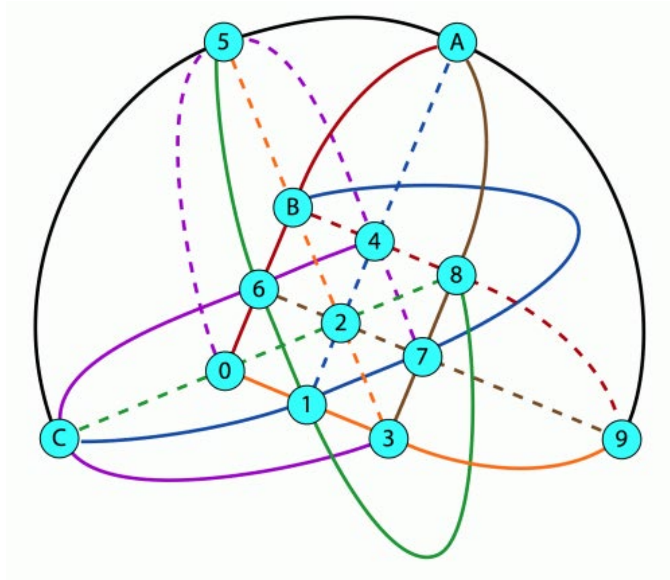
(d) If you go to the webpage

`http://mathworld.wolfram.com/KirkmansSchoolgirlProblem.html`

you'll find an explicit solution to the schoolgirl problem. Justify your answer to part (c) using the explicit solution to the schoolgirl problem found on this webpage. (For example, if your answer to (c) was that there can exist two points that do not lie on a line, go to the webpage and tell me which two of the points $A, B, \ldots, N, O$ do not lie on a common line. I hope this was not your answer to (c)!)

4. (Basically from Moorhouse, pages 36 and 39; the picture is from `http://www.mathpuzzle.com/projplane3.gif`. You will definitely want to view this in color!)

The following is an incidence diagram for the projective plane $\mathbb{P}^2(\mathbb{F}_3)$.



Remember that $\mathbb{F}_3$ is the field with 3 elements, $\{0, 1, 2\}$. So we should be able to label each of the 13 points with a triple $[a, b, c]$ such that $a, b, c \in \mathbb{F}_3$. In this problem I'd like you to do exactly that. The following is a list of the 13 labelled points, and I've put in coordinates for some of them. Please fill in the rest of the table.

Hint 1: remember that the lines in $\mathbb{P}^2(\mathbb{F}_3)$ are all of the form $ax + by + cz = 0$ where $a, b, c \in \mathbb{F}_3$.

Hint 2: remember that lines and points are defined up to non-zero scalar multiples, so for instance $[1, 0, 0]$ is the same point as $[2, 0, 0]$ and the line $x + 2y + z = 0$ is the same as the line $2x + y + 2z = 0$.

Please show all your work and explain systematically how you are getting your answer. I won't give any credit if all I see is a completed table, even if it is correct. Your explanation of your reasoning has to be clear and well-presented.

Point Label	Coordinates
0	$[0, 0, 1]$
1	$[1, 0, 1]$
2	
3	$[2, 0, 1]$
4	
5	
6	$[0, 1, 1]$
7	
8	
9	
A	
B	$[0, 2, 1]$
C	

5. (a) Give an example of two distinct planes in $\mathbb{R}^3$ that do not meet in any points.
   (b) In contrast, explain why two distinct planes in $\mathbb{P}^3(\mathbb{R})$ *have* to meet in a line (as opposed to meeting in a point or not meeting at all).
   (c) Give an example of two distinct *lines* in $\mathbb{P}^3(\mathbb{R})$ that do not meet in any points.

6. Let $\mathbb{P}^2$ be a projective plane (*not necessarily classical*, and not necessarily finite; so all we know is that it satisfies (P1), (P2), (P3)). Let ℓ be a line in $\mathbb{P}^2$. Let $\mathbb{B}$ be the complement of ℓ, i.e. $\mathbb{B}$ is the set of points in $\mathbb{P}^2$ that do not lie on ℓ. Define a *line* in $\mathbb{B}$ to be any set of the form

$$\{m - P \mid m \text{ is a line in } \mathbb{P}^2 \text{ other than } \ell, \text{ and } P = m \cap \ell\}.$$

Prove that $\mathbb{B}$ is an affine plane. That is, prove that it satisfies the following axioms:

(AP1) For any two distinct points A and B in $\mathbb{B}$ there is a unique line in $\mathbb{B}$ containing them.

(AP2) For every line m in $\mathbb{B}$ and point A in $\mathbb{B}$ not on m, there exists a unique line in $\mathbb{B}$ through A and parallel to m.

(AP3) There exist four points in $\mathbb{B}$ such that no three are collinear.